

# $p$ -extended Mathieu series from the Schlömilch series point of view

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**Abstract.** Motivated by certain current results by Parmar and Pogány [9] in which the authors introduced the so-called  $p$ -extended Mathieu series the main aim of this paper is to present a connection between such series and a various types of Schlömilch series.

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## 1. INTRODUCTION AND MOTIVATION

The series of the form

$$S(r) = \sum_{n \geq 1} \frac{2n}{(n^2 + r^2)^2}, \quad r > 0$$

is known in literature as Mathieu series. Émile Leonard Mathieu was the first who investigated such series in 1890 in his book [6]. There is a wide range of various generalizations of the Mathieu series, and one of them is the so-called generalized Mathieu series with a fractional power reads [2, p. 2, Eq. (1.6)] (and also consult [7, p. 181])

$$S_\mu(r) = \sum_{n \geq 1} \frac{2n}{(n^2 + r^2)^{\mu+1}}, \quad r > 0, \mu > 0;$$

which can also be presented in terms of the Riemann Zeta function [2, p. 3, Eq. (2.1)]

$$S_\mu(r) = 2 \sum_{n \geq 0} r^{2n} (-1)^n \binom{\mu + n}{n} \zeta(2\mu + 2n + 1), \quad |r| < 1. \quad (1.1)$$

Having in mind (1.1) Parmar and Pogány [9] recently introduced the  $p$ -extended Mathieu series

$$S_{\mu,p}(r) = 2 \sum_{n \geq 0} r^{2n} (-1)^n \binom{\mu + n}{n} \zeta_p(2\mu + 2n + 1), \quad (1.2)$$

where  $\Re(p) > 0$  or  $p = 0$ ,  $\mu > 0$ . Here and in what follows  $\zeta_p$  stands for the  $p$ -extension of the Riemann  $\zeta$  function [1]:

$$\zeta_p(\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1} e^{-\frac{p}{t}}}{e^t - 1} dt$$

defined for  $\Re(p) > 0$  or  $p = 0$  and  $\Re(\alpha) > 0$  and it reduces to the Riemann zeta function when  $p = 0$ . Also, (1.2) one reduces to (1.1) when  $p = 0$ .

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Parmar and Pogány [9] obtained an integral form of such series, which reads

$$S_{\mu,p}(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu+1)} \int_0^\infty \frac{t^{\mu+\frac{1}{2}}e^{-\frac{p}{t}}}{e^t-1} J_{\mu-\frac{1}{2}}(rt) dt; \quad (1.3)$$

here  $\Re(p) > 0$  or  $p = 0$ ,  $\mu > 0$ .

On the other hand, the series of the form

$$\sum_{n \geq 0} a_n \mathcal{B}_\nu(nz), \quad z \in \mathbb{C},$$

which building blocks  $\mathcal{B}_\nu(\cdot)$  consist from either Bessel, modified Bessel of the first and second kind or alike functions such as Hankel, Lommel, Struve, modified Struve and associated ones of a generic order  $\nu$ , we usually consider under the common name *Schlömilch series* [5, 14].

Motivated by that newly introduced Mathieu series which members contain the extension of the Riemann zeta function  $\zeta_p$  and also the fact that  $\zeta_p$  can be presented as Schlömilch series of modified Bessel functions of the second kind i.e. as [1, p. 1240]

$$\zeta_p(\alpha) = \frac{2p^{\frac{\alpha}{2}}}{\Gamma(\alpha)} \sum_{n \geq 1} \frac{K_\alpha(2\sqrt{np})}{n^{\frac{\alpha}{2}}}, \quad \alpha, p > 0$$

our main aim in this paper is to derive new representations of our series in terms of the various Schlömilch series. In the next section we would derive new representations of (1.2) in terms of Schlömilch series which members contain derivation (ordinary or fractional) of a combination of Bessel function of the first kind  $J_\nu$  and modified Bessel function of the second kind  $K_\nu$ . In the last section we would also derive some connection formulas between our Mathieu series and Schlömilch series but this time with members containing only modified Bessel functions of the second kind.

## 2. CONNECTION BETWEEN $S_{\mu,p}(r)$ AND SCHLÖMILCH SERIES OF $J_\nu \cdot K_\mu$

In this section, our main aim is to derive connection formulas between  $p$ -extended Mathieu series  $S_{\mu,p}(r)$  and Schlömilch series which members contain combination of Bessel functions of the first kind  $J_\nu$  and modified Bessel functions of the second kind  $K_\nu$  of the order  $\nu$ .

Our derivation procedure requires the Grünwald–Letnikov fractional derivative of order  $-\alpha$ ,  $\alpha > 0$  with respect to an argument  $x$  of a suitable function  $f$  defined by [12]

$$\mathbb{D}_x^{-\alpha}[f] = \lim_{n \rightarrow \infty} \left( \frac{n}{x-a} \right)^\alpha \sum_{m=0}^n \frac{\Gamma(\alpha+m)}{m! \Gamma(\alpha)} f \left( x - m \frac{x-a}{h} \right), \quad a < x. \quad (2.1)$$

Several numerical algorithms are available for the direct computation of (2.1); see e.g. [3, 8, 13].

**Theorem 1.** *For all  $\min\{\Re(p), \Re(q), \gamma\} > 0$  and  $\alpha > \frac{1}{2}$  there holds*

$$S_{\alpha-\frac{3}{2},p}(\gamma) = \frac{2(-1)^\alpha \sqrt{\pi}}{(2\gamma)^{\alpha-2} \Gamma(\alpha-\frac{1}{2})} \sum_{k \geq 1} \mathbb{D}_q^\alpha \left( J_{\alpha-2} \left( \sqrt{2p} [\sqrt{q^2 + \gamma^2} - q]^{\frac{1}{2}} \right) \right. \\ \left. \times K_{\alpha-2} \left( \sqrt{2p} [\sqrt{q^2 + \gamma^2} + q]^{\frac{1}{2}} \right) \right) \Big|_{q=k}. \quad (2.2)$$

Further, for  $\alpha = n \in \mathbb{N}$  we have

$$S_{n-\frac{3}{2},p}(\gamma) = \frac{2(-1)^n \sqrt{\pi}}{(2\gamma)^{n-2} \Gamma(n-\frac{1}{2})} \sum_{k \geq 1} \frac{\partial^n}{\partial q^n} \left[ J_{n-2} \left( \sqrt{2p} [\sqrt{q^2 + \gamma^2} - q]^{\frac{1}{2}} \right) \right. \\ \left. \times K_{n-2} \left( \sqrt{2p} [\sqrt{q^2 + \gamma^2} + q]^{\frac{1}{2}} \right) \right] \Big|_{q=k}.$$

*Proof.* In order to prove the desired results, let us first consider the integral [4, p. 708, Eq. 6.635.3]

$$\begin{aligned} A_{p,q}(\gamma) &= \int_0^\infty x^{-1} e^{-qx-p/x} J_\nu(\gamma x) dx \\ &= 2 J_\nu \left( \sqrt{2p} [\sqrt{q^2 + \gamma^2} - q]^{\frac{1}{2}} \right) K_\nu \left( \sqrt{2p} [\sqrt{q^2 + \gamma^2} + q]^{\frac{1}{2}} \right), \end{aligned} \quad (2.3)$$

where  $\min\{\Re(p), \Re(q), \gamma\} > 0$ .<sup>1</sup>

Now, using the Grünwald-Letnikov fractional derivative

$$\mathbb{D}_q^\alpha e^{-qx} = (-x)^\alpha e^{-qx}$$

valid for every real  $\alpha > -\nu$  we get

$$\mathbb{D}_q^\alpha A_{p,q}(\gamma) = (-1)^\alpha \int_0^\infty x^{\alpha-1} e^{-qx-p/x} J_\nu(\gamma x) dx.$$

Further, specifying  $q = k + 1$  and summing up the previous equality for  $k \in \mathbb{N}_0$  we have

$$\sum_{k \geq 0} \mathbb{D}_q^\alpha A_{p,q}(\gamma) \Big|_{q=k+1} = (-1)^\alpha \int_0^\infty \frac{x^{\alpha-1} e^{-p/x}}{e^x - 1} J_\nu(\gamma x) dx.$$

Setting  $\nu = \alpha - 2$  with the help of the integral representation (1.3) we get

$$\int_0^\infty \frac{x^{\alpha-1} e^{-p/x}}{e^x - 1} J_{\alpha-2}(\gamma x) dx = \frac{(2\gamma)^{\alpha-2} \Gamma(\alpha - \frac{1}{2})}{\sqrt{\pi}} S_{\alpha-\frac{3}{2},p}(\gamma)$$

Now, from the previous calculations, using also (2.3), we have

$$\begin{aligned} S_{\alpha-\frac{3}{2},p}(\gamma) &= \frac{(-1)^\alpha \sqrt{\pi}}{(2\gamma)^{\alpha-2} \Gamma(\alpha - \frac{1}{2})} \sum_{k \geq 1} \mathbb{D}_q^\alpha A_{p,q}(\gamma) \Big|_{q=k} \\ &= \frac{2(-1)^\alpha \sqrt{\pi}}{(2\gamma)^{\alpha-2} \Gamma(\alpha - \frac{1}{2})} \sum_{k \geq 1} \mathbb{D}_q^\alpha \left( J_{\alpha-2} \left( \sqrt{2p} [\sqrt{q^2 + \gamma^2} - q]^{\frac{1}{2}} \right) \right. \\ &\quad \left. \times K_{\alpha-2} \left( \sqrt{2p} [\sqrt{q^2 + \gamma^2} + q]^{\frac{1}{2}} \right) \right) \Big|_{q=k} \end{aligned}$$

which is equal to (2.2).

Next, for a positive integer  $\alpha = n$  (in fact  $A_{p,q}(\gamma)$  converges for all  $n + \nu > 0$ ) consider

$$\frac{\partial^n}{\partial q^n} A_{p,q}(\gamma) = (-1)^n \int_0^\infty x^{n-1} e^{-qx-p/x} J_\nu(\gamma x) dx.$$

The same procedure as above yields

$$\int_0^\infty \frac{x^{n-1} e^{-p/x}}{e^x - 1} J_\nu(\gamma x) dx = (-1)^n \sum_{k \geq 0} \frac{\partial^n}{\partial q^n} A_{p,q}(\gamma) \Big|_{q=k+1}.$$

Again with the help of (1.3) and (2.3) and substituting  $\nu = n - 2$  we have

$$\begin{aligned} S_{n-\frac{3}{2},p}(\gamma) &= \frac{(-1)^n \sqrt{\pi}}{(2\gamma)^{n-2} \Gamma(n - \frac{1}{2})} \sum_{k \geq 1} \frac{\partial^n}{\partial q^n} A_{p,q}(\gamma) \Big|_{q=k} = \frac{2(-1)^n \sqrt{\pi}}{(2\gamma)^{n-2} \Gamma(n - \frac{1}{2})} \\ &\quad \times \sum_{k \geq 1} \frac{\partial^n}{\partial q^n} \left[ J_{n-2} \left( \sqrt{2p} [\sqrt{q^2 + \gamma^2} - q]^{\frac{1}{2}} \right) K_{n-2} \left( \sqrt{2p} [\sqrt{q^2 + \gamma^2} + q]^{\frac{1}{2}} \right) \right] \Big|_{q=k}, \end{aligned}$$

which completes the proof.  $\square$

<sup>1</sup>Actually,  $A_{p,q}(\gamma)$  is the Laplace transform of  $x \mapsto x^{-1} e^{-p/x} J_\nu(\gamma x)$  at the argument  $q$ .

**Theorem 2.** For all  $\Re(p) > 0$  we have

$$S_{\frac{1}{2},p}(\gamma) = -4\gamma \sum_{k \geq 1} \frac{\partial^3}{\partial q^3} \left( \frac{J_1(\sqrt{2p}[\sqrt{q^2 + \gamma^2} - q]^{\frac{1}{2}})K_0(\sqrt{2p}[\sqrt{q^2 + \gamma^2} + q]^{\frac{1}{2}})}{\sqrt{2p}[\sqrt{q^2 + \gamma^2} + q]^{\frac{1}{2}}} \right. \\ \left. + \frac{J_0(\sqrt{2p}[\sqrt{q^2 + \gamma^2} - q]^{\frac{1}{2}})K_1(\sqrt{2p}[\sqrt{q^2 + \gamma^2} + q]^{\frac{1}{2}})}{\sqrt{2p}[\sqrt{q^2 + \gamma^2} - q]^{\frac{1}{2}}} \right) \Big|_{q=k}. \quad (2.4)$$

Moreover, it is

$$S_{-\frac{1}{2},p}(\gamma) = 4\gamma \sum_{k \geq 1} \frac{\partial}{\partial q} \left( J_1(\sqrt{2p}[\sqrt{q^2 + \gamma^2} - q]^{\frac{1}{2}})K_1(\sqrt{2p}[\sqrt{q^2 + \gamma^2} + q]^{\frac{1}{2}}) \right) \Big|_{q=k}. \quad (2.5)$$

*Proof.* With the help of the integral [10, p. 188, Eq. **2.12.10.2**]

$$B_{p,q}(\gamma) = \int_0^\infty x^{-2} e^{-qx-p/x} J_0(\gamma x) dx = 2\gamma (z_+^{-1} J_1(z_-) K_0(z_+) + z_-^{-1} J_0(z_-) K_1(z_+)),$$

where  $z_\pm = \sqrt{2p}[\sqrt{q^2 + \gamma^2} \pm q]^{1/2}$ ,  $\min\{\Re(q), \Re(p)\} > 0$ , we conclude

$$\frac{\partial^3}{\partial q^3} B_{p,q}(\gamma) = - \int_0^\infty x e^{-qx-p/x} J_0(\gamma x) dx$$

which, with the help of (1.3), gives us

$$\sum_{k \geq 0} \frac{\partial^3}{\partial q^3} B_{p,q}(\gamma) \Big|_{q=k+1} = - \int_0^\infty \frac{x e^{-p/x}}{e^x - 1} J_0(\gamma x) dx = -\frac{1}{2} S_{1/2,p}(\gamma)$$

which coincides with (2.4).

In the same way, but this time using [10, p. 188, Eq. **2.12.10.1**]

$$C_{p,q}(\gamma) = \int_0^\infty x^{-1} e^{-qx-p/x} J_\nu(\gamma x) dx = 2J_\nu(z_-)K_\nu(z_+),$$

where  $\min\{\Re(p), \Re(q)\} > 0$ , and  $z_\pm$  has the same meaning as above, with the aid of parity of Bessel and modified Bessel function  $J_{-1}(x) = -J_1(x)$ ;  $K_{-1}(x) = K_1(x)$  we deduce (2.5).  $\square$

**Remark 1.** From (2.5), bearing in mind [15, 16]:

$$2(J_1(x)K_1(x))' = (J_0(x) - J_2(x))K_1(x) - J_1(x)(K_0(x) + K_2(x)),$$

we can infer a new representation for  $S_{-\frac{1}{2},p}(\gamma)$ .

### 3. $S_{\mu,p}(r)$ AND THE SCHLÖMILCH SERIES OF $K_\nu$ TERMS

Considering now specialized  $p$ -extended Mathieu series, that is in which  $\mu = 0, 1, 2$ , we report on their Schlömilch-series expansion *via* modified Bessel functions of the second kind  $K_{\mu+1}$ .

**Theorem 3.** For all  $\Re(p) > 0$ ,  $\gamma > 0$  there hold

$$S_{0,p}(\gamma) = 2\sqrt{p} \sum_{k \geq 1} \left( \frac{K_1(2\sqrt{p(k+i\gamma)})}{\sqrt{k+i\gamma}} + \frac{K_1(2\sqrt{p(k-i\gamma)})}{\sqrt{k-i\gamma}} \right), \quad (3.1)$$

$$S_{1,p}(\gamma) = \frac{p i}{\gamma} \sum_{k \geq 1} \left( \frac{K_2(2\sqrt{p(k+i\gamma)})}{\sqrt{k+i\gamma}} - \frac{K_2(2\sqrt{p(k-i\gamma)})}{\sqrt{k-i\gamma}} \right). \quad (3.2)$$

*Proof.* In order to prove the desired results we will need the following formula [11]

$$\begin{aligned} E_{p,q}^{\mp}(\gamma) &= \int_0^{\infty} x^{\nu} e^{-qx-p/x} \left\{ \frac{\sin(\gamma x)}{\cos(\gamma x)} \right\} dx \\ &= i^{\frac{1+\nu}{2}} p^{\frac{\nu+1}{2}} \left( \frac{K_{\nu+1} \left( 2\sqrt{p(q+i\gamma)} \right)}{(q+i\gamma)^{\frac{\nu+1}{2}}} \mp \frac{K_{\nu+1} \left( 2\sqrt{p(q-i\gamma)} \right)}{(q-i\gamma)^{\frac{\nu+1}{2}}} \right), \end{aligned} \quad (3.3)$$

which holds for  $\min\{\Re(p), \Re(q)\} > 0$ .

Now, since

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x,$$

by virtue of (1.3) and (3.3) setting  $q = k + 1$ ,  $k \in \mathbb{N}_0$  and  $\nu = 0$  it follows

$$\sum_{k \geq 0} E_{p,k+1}^+(\gamma) = \sqrt{\frac{\pi\gamma}{2}} \int_0^{\infty} \frac{\sqrt{x} e^{-p/x}}{e^x - 1} J_{-\frac{1}{2}}(\gamma x) dx = \frac{1}{2} S_{0,p}(\gamma),$$

which results in (3.1).

Analogously, from (1.3) for  $\nu = 1$ , applying (3.3) for  $E_{p,k+1}^-(\gamma)$  and  $J_{\frac{1}{2}}(x) = \sqrt{2/(\pi x)} \sin x$ , one implies the second statement (3.2).  $\square$

**Theorem 4.** For all  $\Re(p) > 0$ ,  $r > 0$  there holds

$$S_{2,p}(r) = \frac{1}{(2r)^2} S_{1,p}(r) - \frac{p^{\frac{3}{2}}}{(2r)^2} \sum_{n \geq 1} \left( \frac{K_3 \left( 2\sqrt{p(n+i r)} \right)}{(n+i r)^{\frac{3}{2}}} + \frac{K_3 \left( 2\sqrt{p(n-i r)} \right)}{(n-i r)^{\frac{3}{2}}} \right). \quad (3.4)$$

*Proof.* From the integral representation (1.3) of  $S_{\mu,p}(r)$ , for  $\mu = 2$  it is

$$\begin{aligned} S_{2,p}(r) &= \frac{\sqrt{\pi}}{2(2r)^{3/2}} \int_0^{\infty} \frac{x^{5/2} e^{-p/x} J_{\frac{3}{2}}(rx)}{e^x - 1} dx \\ &= \frac{\sqrt{\pi}}{2(2r)^{3/2}} \sum_{k \geq 1} \int_0^{\infty} x^{5/2} e^{-kx-p/x} J_{\frac{3}{2}}(rx) dx \\ &= \frac{1}{(2r)^2} \sum_{k \geq 1} \int_0^{\infty} x^2 e^{-kx-p/x} \left( \frac{\sin(rx)}{rx} - \cos(rx) \right) dx, \end{aligned}$$

where in the last equality we used the well-known formula

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left( \frac{\sin x}{x} - \cos x \right).$$

Further, with the help of (1.3) and  $J_{\frac{1}{2}}(x) = \sqrt{2/(\pi x)} \sin x$  the previous expression can be rewritten into

$$S_{2,p}(r) = \frac{1}{(2r)^2} S_{1,p}(r) - \frac{1}{(2r)^2} \sum_{k \geq 1} \int_0^{\infty} x^2 e^{-kx-p/x} \cos(rx) dx.$$

Finally, using the Laplace transform of the function  $x \mapsto x^2 e^{-p/x} \cos(rx)$ , in the argument  $k$  given by (3.3) we get the display (3.4).  $\square$

**Remark 2.** Using the formula (3.2) derived in Theorem 3 and the formula (3.4) which connects  $S_{2,p}(r)$  and  $S_{1,p}(r)$  new representation for  $S_{2,p}(r)$  can be derived.

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